

resonance production in two-photon physics

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Abstract

In this paper a Monte Carlo generator, GaGaRes, is presented which can be used to describe two-photon resonance production in e^+e^- collisions. The program can generate the five lowest-lying $C=+1$ meson states of any $q\bar{q}$ combination together with the outgoing electron and positron. The dependence on the photon virtualities Q_1^2 and Q_2^2 is fully taken into account. The program also generates the density matrices of the resonance, which form an essential tool in the description of the decay of the resonances. Furthermore, the program is applicable for all tagging conditions.

PROGRAM SUMMARY

Title of program: GaGaRes

Program obtainable from: CPC Library

Computer for which the program is designed and others on which it is operable: Any computer that runs under UNIX and can handle quadruple precision numbers (both real and complex). FORTRAN 90 offers the possibility to define own data types. This should be used on systems where complex numbers are not available in quadruple precision.

Operating system under which the program has been tested : UNIX

Programming language used : FORTRAN 90

Number of lines: 6009 (including comments)

Keywords: Monte Carlo, two-photon, e^+e^- , resonance production

Nature of physical problem: With the advent of LEP2 higher energies for two-photon reactions became available with high luminosities. This makes it possible to search experimentally for heavier resonances created in two-photon collisions and also to determine the dependence of the two-photon cross sections on the virtualities Q_1^2 and Q_2^2 . Moreover, the decay distributions of the resonances can be studied. These experimental possibilities make it desirable to have a program, which can simulate events as expected from our theoretical understanding of resonance production by two photons.

Methods of solution: A model based on the hard scattering approach is used to describe the production of the resonances[1]. For an exact description of the decay of the produced resonance the density matrix is required. Weyl-van-der-Waerden spinor calculations are used to obtain these density matrices. Events consisting of the momenta of the resonance and the outgoing electron and positron are generated by Monte Carlo methods and are distributed according to the theoretical cross section.

Typical running time: Depends on the requested accuracy and the generated resonance. On an Enterprise450, one can produce 380 1S_0 resonances/sec and 17 3P_2 resonances/sec, including the corresponding density matrix. Without density matrices, one can generate 236 3P_2 resonances/sec. These numbers are for weighted events. For unweighted events these rates are typically an order of magnitude lower.

1 Introduction

With the advent of LEP2 a new region for two-photon physics was entered. Its high energy and its high luminosity, resulting in better statistics, allows one to look for the dependence of the two-photon cross sections on the $\gamma\gamma$ centre-of-mass energy W and on the photon virtualities Q_1^2 and Q_2^2 . The higher statistics also gives one the opportunity to look for the decay distributions of the produced resonances.

The theoretical framework [1] on which the program is based is obtained from the hard scattering approach in which the meson resonance is treated as a non-relativistic bound state of a quark (q) and an anti-quark (\bar{q}). In order to include also the production of lighter resonances some modifications had to be implemented. These modifications are discussed in [1]. The two-photon widths of the resonances that are calculated using this model give a reasonably good agreement with experiment, also for the lighter resonances.

The matrix elements following from this model have been implemented in the program, including all dependences on W and Q_i^2 . This is in contrast to most other two-photon MC generators [2, 3] where the Q_i^2 dependence is ad-hoc parametrized by multiplying the cross section by the square of a form factor $F^2(Q_1^2, Q_2^2)$. The VMD model predicts the pole-mass form factor

$$F(Q_1^2, Q_2^2) = \frac{M_V^2}{(M_V^2 + Q_1^2)} \frac{M_V^2}{(M_V^2 + Q_2^2)}. \quad (1)$$

In this form factor the scaling mass M_V is the mass of a resonance. For the lighter resonances one takes $V = \rho$, for the $c\bar{c}$ resonances $V = J/\psi$ whereas for the $b\bar{b}$ resonances $V = \Upsilon$. Note that this form factor causes a very strong decrease of the cross section with high Q_i^2 's. In [1] it is discussed that the model used in GaGaRes results in a much weaker decrease of the cross section with high Q_i^2 .

The total angular momentum of the resonance is denoted by J , its spin by $S(=0, 1)$ and its orbital angular momentum by $L(=0, 1, 2, \dots$ or $S, P, D, \dots)$, resulting in the well-known spectroscopic notation $^{(2S+1)}L_J$, with parity $P = (-1)^{L+1}$ and charge conjugation $C = (-1)^{L+S}$.

In two-photon reactions one can, in lowest order, only produce C -even resonances. The present version of GaGaRes describes the production of 1S_0 (0^-), 3P_J ($J = 0, 1, 2$) (J^+) and 1D_2 (2^-) resonances. The resonance can be composed of light (u, d, s) quarks in the states $I = 0, 1$, where I is the isospin, or of heavy c and b quarks.

The outline of the paper is as follows. We start by introducing the kinematics and notation and present the different calculational methods to obtain the expressions for the matrix elements squared. Then the discussion focusses on the density matrices, which are required to describe the decay of the resonances. This is followed by a description of the two Monte Carlo generation schemes available in GaGaRes for the integration of the differential cross section. Finally, the structure of the program is presented, together with some important variables, common blocks and procedures.

2 Notation, Kinematics and Calculations of the Matrix Elements Squared

The process of resonance production in two-photon physics is described by the following reaction

$$\begin{aligned} e^+(p_1) + e^-(p_2) &\rightarrow e^+(p_3) + e^-(p_4) + \gamma^*(k_1) + \gamma^*(k_2) \\ &\rightarrow e^+(p_3) + e^-(p_4) + R(P). \end{aligned} \quad (2)$$

In this reaction the γ^* 's are off-shell photons which collide to form the resonance. Four-momentum conservation in the vertices gives

$$\begin{aligned} k_1 &= p_1 - p_3, \\ k_2 &= p_2 - p_4, \\ P &= k_1 + k_2. \end{aligned} \quad (3)$$

The following parametrization of the external four-momenta in the lab-frame has been chosen.

$$\begin{aligned} p_1 &= (E_b, 0, 0, -P_b), \\ p_2 &= (E_b, 0, 0, P_b), \\ p_3 &= (E_3, |\vec{p}_3| \sin \theta_3 \cos \phi_3, |\vec{p}_3| \sin \theta_3 \sin \phi_3, -|\vec{p}_3| \cos \theta_3), \\ p_4 &= (E_4, |\vec{p}_4| \sin \theta_4 \cos \phi_4, |\vec{p}_4| \sin \theta_4 \sin \phi_4, |\vec{p}_4| \cos \theta_4), \\ P &= (E_R, |\vec{p}_R| \sin \theta_R \cos \phi_R, |\vec{p}_R| \sin \theta_R \sin \phi_R, |\vec{p}_R| \cos \theta_R). \end{aligned} \quad (4)$$

In these formulae E_b is the beam-energy. P_b is the momentum of the incoming leptons. The polar angles θ_3 and θ_4 are defined as the angle between the incoming lepton and the corresponding outgoing lepton.

It is convenient to introduce the set of invariants

$$\begin{aligned} s &= (p_1 + p_2)^2, \\ s' &= (p_3 + p_4)^2, \\ t &= k_1^2 = -Q_1^2 = (p_1 - p_3)^2, \\ t' &= k_2^2 = -Q_2^2 = (p_2 - p_4)^2, \\ u &= (p_1 - p_4)^2, \\ u' &= (p_2 - p_3)^2, \\ W^2 &= P^2 = (k_1 + k_2)^2, \end{aligned} \quad (5)$$

where Q_1^2 and Q_2^2 are called the virtualities of the intermediate photons. These invariants satisfy the relation

$$s + s' + t + t' + u + u' = W^2 + 8m_e^2. \quad (6)$$

Using the methods of [1] expressions for the matrix elements of reaction (2) can be

obtained¹

$$\begin{aligned}
\mathcal{M}(^1S_0; \lambda_1, \lambda_2, \lambda_3, \lambda_4) &= \frac{c_1}{tt'} \varepsilon [k_1, k_2, j_1(\lambda_1, \lambda_3), j_2(\lambda_2, \lambda_4)], \\
\mathcal{M}(^3P_0; \lambda_1, \lambda_2, \lambda_3, \lambda_4) &= \frac{c_2}{tt'} ([j_1(\lambda_1, \lambda_3) \cdot j_2(\lambda_2, \lambda_4) k_1 \cdot k_2 \\
&\quad - j_1(\lambda_1, \lambda_3) \cdot k_2 j_2(\lambda_2, \lambda_4) \cdot k_1] (W^2 + k_1 \cdot k_2) \\
&\quad - j_1(\lambda_1, \lambda_3) \cdot j_2(\lambda_2, \lambda_4) tt'), \\
\mathcal{M}(^3P_1; \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_R) &= \frac{c_3}{tt'} (t\varepsilon [\varepsilon^*(\lambda_R), j_1(\lambda_1, \lambda_3), j_2(\lambda_2, \lambda_4), k_2] \\
&\quad + t'\varepsilon [\varepsilon^*(\lambda_R), j_2(\lambda_2, \lambda_4), j_1(\lambda_1, \lambda_3), k_1]), \\
\mathcal{M}(^3P_2; \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_R) &= \frac{c_4}{tt'} (k_1 \cdot k_2 j_{1\mu}(\lambda_1, \lambda_3) j_{2\nu}(\lambda_2, \lambda_4) \\
&\quad + k_{1\mu} k_{2\nu} j_1(\lambda_1, \lambda_3) \cdot j_2(\lambda_2, \lambda_4) \\
&\quad - k_{1\mu} j_{2\nu}(\lambda_2, \lambda_4) j_1(\lambda_1, \lambda_3) \cdot k_2 \\
&\quad - k_{2\mu} j_{1\nu}(\lambda_1, \lambda_3) j_2(\lambda_2, \lambda_4) \cdot k_1) \varepsilon^{*\mu\nu}(\lambda_R), \\
\mathcal{M}(^1D_2; \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_R) &= \frac{c_5}{tt'} \varepsilon^{*\mu\nu}(\lambda_R) k_{1\mu} k_{2\nu} \\
&\quad \varepsilon [k_1, k_2, j_1(\lambda_1, \lambda_3), j_2(\lambda_2, \lambda_4)].
\end{aligned} \tag{7}$$

In these formulae λ_i represents the helicity of the associated external particle. The lepton currents j_1 and j_2 are defined by

$$\begin{aligned}
j_1^\mu(\lambda_1, \lambda_3) &= \bar{v}_{\lambda_1}(p_1) \gamma^\mu v_{\lambda_3}(p_3), \\
j_2^\mu(\lambda_2, \lambda_4) &= \bar{u}_{\lambda_4}(p_4) \gamma^\mu u_{\lambda_2}(p_2),
\end{aligned} \tag{8}$$

satisfying current-conservation:

$$j_i(\lambda_m, \lambda_n) \cdot k_i = 0 \quad (i = 1, 2). \tag{9}$$

The polarization vector of the outgoing spin-1 resonance with helicity λ_R is $\varepsilon^{\mu*}(\lambda_R)$ whereas $\varepsilon^{\mu\nu*}(\lambda_R)$ is the polarization tensor of the outgoing spin-2 resonances with helicity λ_R . We use the polarization vectors as defined in (3.15) of [4]. With this choice, the spin-2 polarization tensors can be constructed from the spin-1 polarization vectors using the Clebsch-Gordan coefficients

$$\begin{aligned}
\varepsilon^{*\mu\nu}(\pm 2) &= \varepsilon^{*\mu}(\pm 1) \varepsilon^{*\nu}(\pm 1), \\
\varepsilon^{*\mu\nu}(\pm 1) &= \pm \frac{1}{\sqrt{2}} (\varepsilon^{*\mu}(\pm 1) \varepsilon^{*\nu}(0) + \varepsilon^{*\mu}(0) \varepsilon^{*\nu}(\pm 1)), \\
\varepsilon^{*\mu\nu}(0) &= \frac{1}{\sqrt{6}} (-\varepsilon^{*\mu}(+1) \varepsilon^{*\nu}(-1) + 2\varepsilon^{*\mu}(0) \varepsilon^{*\nu}(0) - \varepsilon^{*\mu}(-1) \varepsilon^{*\nu}(1)).
\end{aligned} \tag{10}$$

Finally, the constants c_1, \dots, c_5 are given by

$$c_1 = g_0, \quad c_2 = 4g_1/W, \quad c_3 = 2\sqrt{6}g_1, \quad c_4 = 4\sqrt{3}Wg_1, \quad c_5 = 8\sqrt{30}g_2, \tag{11}$$

where we have introduced

$$g_i = \frac{16e^2 e_q^2 |\mathcal{R}^{(i)}(0)| \alpha}{(s + s' + u + u' - 8m_e^2)^{i+1}} \sqrt{\frac{3\pi}{W}}. \tag{12}$$

The values used for the fractional charges e_q^2 and for (the derivatives of) the radial part of the wave functions in the origin, $|R(0)|$, can be found in [1].

The above formulae lead to expressions and numerical results for the total matrix element squared and the density matrices. We thus follow a calculational strategy that

¹For the Minkowski metric we use $(1, -1, -1, -1)$. The totally antisymmetric Levi-Civita tensor is defined by $\varepsilon_{0123} = +1$. Also the notation $\varepsilon[p, q, r, s] \equiv \varepsilon^{\mu\nu\lambda\rho} p_\mu q_\nu r_\lambda s_\rho$ is introduced.

differs from the usual one [5], in which the expression for the total cross section is split into three parts: two density matrices for the production of the virtual photons from the external leptons and a cross section for the process $\gamma^*\gamma^* \rightarrow R$, which in turn can be written in the well-known form

$$\begin{aligned} d\sigma = & \frac{\alpha^2}{16\pi^4 k_1^2 k_2^2} \sqrt{\frac{X_{\gamma\gamma}}{X_{e^+e^-}}} \left\{ 4\rho_1^{++}\rho_2^{++}\sigma_{TT} + 2\rho_1^{++}\rho_2^{00}\sigma_{TS} \right. \\ & + 2\rho_1^{00}\rho_2^{++}\sigma_{ST} + \rho_1^{00}\rho_2^{00}\sigma_{SS} + 2|\rho_1^{+-}\rho_2^{+-}|\tau_{TT}\cos(2\tilde{\phi}) \\ & \left. - 8|\rho_1^{+0}\rho_2^{+0}|\tau_{TS}\cos(\tilde{\phi}) \right\} \frac{d^3\vec{p}_3}{E_3} \frac{d^3\vec{p}_4}{E_4}, \end{aligned} \quad (13)$$

where $\tilde{\phi}$ is the angle between the two scattering planes in the $\gamma\gamma$ rest frame.

In GaGaRes there are three methods implemented for the calculation of the square of the matrix element summed over the helicities of the external particles.

In the first method we use an expression for the total matrix element squared in terms of the invariants introduced in (5). These expressions have been obtained using FORM [6]. In the calculation we have introduced the tensors formed by the product of a lepton-current with its complex conjugated, summed over all helicities

$$\begin{aligned} L_1^{\mu\nu} &= \sum_{\lambda_1, \lambda_3} j_1^\mu(\lambda_1, \lambda_3) j_1^{*\nu}(\lambda_1, \lambda_3) = 4(p_1^\mu p_3^\nu + p_3^\mu p_1^\nu + \frac{1}{2}t g^{\mu\nu}), \\ L_2^{\mu\nu} &= \sum_{\lambda_2, \lambda_4} j_2^\mu(\lambda_2, \lambda_4) j_2^{*\nu}(\lambda_2, \lambda_4) = 4(p_2^\mu p_4^\nu + p_4^\mu p_2^\nu + \frac{1}{2}t' g^{\mu\nu}). \end{aligned} \quad (14)$$

We also use the standard summation rules for the polarization vectors/tensors of massive particles

$$\begin{aligned} \sum_{\lambda_R} \varepsilon^\mu(\lambda_R) \varepsilon^{*\nu}(\lambda_R) &= -g^{\mu\nu} + \frac{P_\mu P_\nu}{M^2} \equiv \mathcal{P}_{\mu\nu}, \\ \sum_{\lambda_R} \varepsilon^{\mu\nu}(\lambda_R) \varepsilon^{*\rho\sigma}(\lambda_R) &= \frac{1}{2}(\mathcal{P}_{\mu\rho} \mathcal{P}_{\nu\sigma} + \mathcal{P}_{\mu\sigma} \mathcal{P}_{\nu\rho}) - \frac{1}{3} \mathcal{P}_{\mu\nu} \mathcal{P}_{\rho\sigma}. \end{aligned} \quad (15)$$

The results of this exercise are collected in appendix A. The numerical calculation of the total cross section involving these expressions is the fastest method in GaGaRes.

We now turn to the two other calculational methods, which are options in the program. They are based on the evaluation of helicity amplitudes in the Weyl-van-der-Waerden (WvdW) spinor formalism [4, 7]. These methods are slower than the method using the expression in terms of invariants, but they allow a straightforward construction of density matrices for the produced resonances. In the WvdW calculations we use the notation and conventions introduced in [4]. Furthermore the Levi-Civita tensor can be written in the WvdW formalism as [8]

$$\varepsilon_{\dot{A}W\dot{B}X\dot{C}Y\dot{D}Z} = 4i(\varepsilon_{\dot{A}\dot{B}}\varepsilon_{\dot{C}\dot{D}}\varepsilon_{WY}\varepsilon_{XZ} - \varepsilon_{\dot{A}\dot{C}}\varepsilon_{\dot{B}\dot{D}}\varepsilon_{WX}\varepsilon_{YZ}), \quad (16)$$

where ε_{AB} is the metric tensor for the Weyl spinors, given by

$$\varepsilon_{AB} = \varepsilon_{\dot{A}\dot{B}} = \varepsilon^{AB} = \varepsilon^{\dot{A}\dot{B}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (17)$$

This metric tensor also defines the (antisymmetric) inner product of two Weyl spinors

$$<pk> = \varepsilon^{AB} p_A k_B = p_A k^A = -<kp>, \quad <pp> = 0. \quad (18)$$

In the WvdW formalism each Minkowski four-vector k^μ can be related to a WvdW bispinor

$$k^\mu = (k^0, k^1, k^2, k^3) \leftrightarrow K_{\dot{A}B} = \sigma_{\mu, \dot{A}B} k^\mu = \begin{pmatrix} k^0 + k^3 & k^1 + ik^2 \\ k^1 - ik^2 & k^0 - k^3 \end{pmatrix}. \quad (19)$$

As a consequence the inner products of two Minkowski four-vectors can be written as

$$k \cdot p = \frac{1}{2} K_{\dot{A}B} P^{\dot{A}B} = \frac{1}{2} K^{\dot{A}B} P_{\dot{A}B}, \quad k^2 \delta_B^C = K_{\dot{A}B} K^{\dot{A}C}. \quad (20)$$

A WvdW bispinor corresponding to a time-like ($k^2 > 0$) or light-like ($k^2 = 0$) Minkowski four-vector $k^\mu = (k^0, |\vec{k}| \sin \theta \cos \phi, |\vec{k}| \sin \theta \sin \phi, |\vec{k}| \cos \theta)$ can be decomposed into two Weyl spinors

$$K_{\dot{A}B} = \sum_{i=1,2} \kappa_{i,\dot{A}} \kappa_{i,B}, \quad (21)$$

with

$$\kappa_{1,A} = \sqrt{k^0 + |\vec{k}|} \begin{pmatrix} e^{-i\phi} \cos \frac{\theta}{2} \\ \sin \frac{\theta}{2} \end{pmatrix}, \quad \kappa_{2,A} = \sqrt{k^0 - |\vec{k}|} \begin{pmatrix} \sin \frac{\theta}{2} \\ -e^{i\phi} \cos \frac{\theta}{2} \end{pmatrix}. \quad (22)$$

From these formulae one sees that for light-like four-vectors the WvdW expressions simplify dramatically as for these four-vectors the Weyl spinors with subindex 2 vanish. For these four-vectors the corresponding WvdW bispinor can be written in a simple dyad form.

The usual Dirac spinors for helicity $\pm \frac{1}{2}$ particles and antiparticles can also be expressed in the above Weyl spinors

$$u_+ = \begin{pmatrix} \kappa_{1,A} \\ -\kappa_{2,\dot{A}} \end{pmatrix}, \quad u_- = \begin{pmatrix} \kappa_{2,A} \\ \kappa_{1,\dot{A}} \end{pmatrix}, \quad v_+ = \begin{pmatrix} -\kappa_{2,A} \\ \kappa_{1,\dot{A}} \end{pmatrix}, \quad v_- = \begin{pmatrix} \kappa_{1,A} \\ \kappa_{2,\dot{A}} \end{pmatrix}, \quad (23)$$

$$\bar{u}_+ = (-\kappa_{2,A}^A, \kappa_{1,\dot{A}}), \quad \bar{u}_- = (\kappa_{1,A}^A, \kappa_{2,\dot{A}}), \quad \bar{v}_+ = (\kappa_{1,A}^A, -\kappa_{2,\dot{A}}), \quad \bar{v}_- = (\kappa_{2,A}^A, \kappa_{1,\dot{A}}). \quad (24)$$

Using

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma_{\dot{B}A}^\mu \\ \sigma_{\dot{A}B}^\mu & 0 \end{pmatrix}, \quad (25)$$

the currents (8) can also be written in the WvdW formalism, for positrons one has

$$\begin{aligned} J_1^{\dot{R}S}(++) &= 2 \left[(p_3)_1^{\dot{R}} (p_1)_1^S + (p_1)_2^{\dot{R}} (p_3)_2^S \right], \\ J_1^{\dot{R}S}(+-) &= 2 \left[(p_3)_2^{\dot{R}} (p_1)_1^S - (p_1)_2^{\dot{R}} (p_3)_1^S \right], \\ J_1^{\dot{R}S}(-+) &= 2 \left[(p_3)_1^{\dot{R}} (p_1)_2^S - (p_1)_1^{\dot{R}} (p_3)_2^S \right], \\ J_1^{\dot{R}S}(--) &= 2 \left[(p_1)_1^{\dot{R}} (p_3)_1^S + (p_3)_2^{\dot{R}} (p_1)_2^S \right]. \end{aligned} \quad (26)$$

The currents for the electron line can be obtained from these currents by making the substitutions $p_1 \leftrightarrow p_2$ and $p_3 \leftrightarrow p_4$.

It can be checked that these WvdW bispinors satisfy current-conservation, which reads in WvdW notation

$$J_i^{\dot{A}B}(\lambda_m, \lambda_n) K_{i,\dot{A}B} = 0 \quad (i = 1, 2). \quad (27)$$

Finally, the polarization tensors for the massive spin-2 resonances can be constructed from the spin-1 polarization vectors using the Clebsch-Gordan coefficients given in (10).

The expressions for the amplitudes in (7) read in the WvdW notation

$$\begin{aligned}
\mathcal{M}(^1S_0)(\lambda_1, \lambda_2, \lambda_3, \lambda_4) &= \frac{c_1}{4tt'} \left\{ J_{1,\dot{C}Y}(\lambda_1, \lambda_3) K_1^{\dot{B}Y} K_{2,\dot{B}Z} J_2^{\dot{C}Z}(\lambda_2, \lambda_4) \right. \\
&\quad \left. - J_{1,\dot{C}Y}(\lambda_1, \lambda_3) K_1^{\dot{C}X} K_{2,\dot{D}X} J_2^{\dot{D}Y}(\lambda_2, \lambda_4) \right\}, \\
\mathcal{M}(^3P_0)(\lambda_1, \lambda_2, \lambda_3, \lambda_4) &= \frac{c_2}{4tt'} \left\{ 2H J_{1,\dot{A}B}(\lambda_1, \lambda_3) J_2^{\dot{A}B}(\lambda_2, \lambda_4) \right. \\
&\quad \left. - G J_{1,\dot{A}B}(\lambda_1, \lambda_3) J_{2,\dot{C}D}(\lambda_2, \lambda_4) K_2^{\dot{A}B} K_1^{\dot{C}D} \right\}, \\
\mathcal{M}(^3P_1)(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_R) &= \frac{c_3}{4tt'} \left\{ t [J_{2,\dot{C}Y}(\lambda_2, \lambda_4) \varepsilon^{*\dot{B}Y}(\lambda_R) J_{1,\dot{B}Z}(\lambda_1, \lambda_3) K_2^{\dot{C}Z} \right. \\
&\quad - J_{2,\dot{C}Y}(\lambda_2, \lambda_4) \varepsilon^{*\dot{C}X}(\lambda_R) J_{1,\dot{D}X}(\lambda_1, \lambda_3) K_2^{\dot{D}Y}] \\
&\quad + t' [J_{1,\dot{C}Y}(\lambda_1, \lambda_3) \varepsilon^{*\dot{B}Y}(\lambda_R) J_{2,\dot{B}Z}(\lambda_2, \lambda_4) K_1^{\dot{C}Z} \\
&\quad - J_{1,\dot{C}Y}(\lambda_1, \lambda_3) \varepsilon^{*\dot{C}X}(\lambda_R) J_{2,\dot{D}X}(\lambda_2, \lambda_4) K_1^{\dot{D}Y}] \left. \right\}, \quad (28) \\
\mathcal{M}(^3P_2)(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_R) &= \frac{c_4}{8tt'} \left\{ 2F J_{1,\dot{A}B}(\lambda_1, \lambda_3) J_{2,\dot{C}D}(\lambda_2, \lambda_4) \right. \\
&\quad + K_{1,\dot{A}B} K_{2,\dot{C}D} J_{1,\dot{E}F}(\lambda_1, \lambda_3) J_2^{\dot{E}F}(\lambda_2, \lambda_4) \\
&\quad - K_{1,\dot{A}B} J_{2,\dot{C}D}(\lambda_2, \lambda_4) J_{1,\dot{E}F}(\lambda_1, \lambda_3) K_2^{\dot{E}F} \\
&\quad - K_{2,\dot{A}B} J_{1,\dot{C}D}(\lambda_1, \lambda_3) J_{2,\dot{E}F}(\lambda_2, \lambda_4) K_1^{\dot{E}F} \left. \right\} \\
&\quad \varepsilon^{*\dot{A}\dot{B}\dot{C}\dot{D}}(\lambda_R), \\
\mathcal{M}(^1D_2)(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_R) &= \frac{c_5}{4c_1} \left\{ K_{1,\dot{A}B} K_{2,\dot{C}D} \varepsilon^{*\dot{A}\dot{B}\dot{C}\dot{D}}(\lambda_R) \right\} \\
&\quad \mathcal{M}(^1S_0)(\lambda_1, \lambda_2, \lambda_3, \lambda_4),
\end{aligned}$$

where

$$\begin{aligned}
F &= k_1 \cdot k_2, \\
G &= W^2 + F, \\
H &= FG - tt'.
\end{aligned} \quad (29)$$

The second and third calculational methods in GaGaRes both use the WvdW calculation. In the second method the matrix elements are finally expressed in terms of the standard spinor inner products as defined in equation (18). For the derivation of these expressions FORM has been used.

The third method expresses the matrix element in traces of 2×2 matrices. Let us introduce a down-matrix K_\downarrow

$$K_\downarrow \equiv K_{\dot{A}B}, \quad (30)$$

and an up-matrix K^\uparrow

$$K^\uparrow \equiv (K_\uparrow)^T, \quad K_\uparrow \equiv K^{\dot{A}B}. \quad (31)$$

With these matrices a product of an even number of WvdW bispinors can be rewritten as the trace of a product of the previously defined down- and up-matrices, e.g.

$$\begin{aligned}
P^{\dot{A}B} K_{\dot{A}B} &= \text{Tr} [P^\uparrow K_\downarrow] = 2p \cdot k, \\
P_{\dot{A}B} Q^{\dot{C}B} R_{\dot{C}D} S^{\dot{A}D} &= \text{Tr} [P_\downarrow Q^\uparrow R_\downarrow S^\uparrow].
\end{aligned} \quad (32)$$

One also needs the expressions for the spin-2 polarization vectors in this method

$$\begin{aligned}
\text{Tr} [P^\uparrow \varepsilon_{\downarrow}^*(\pm 2) K^\uparrow] &= \text{Tr} [P^\uparrow \varepsilon_{\downarrow}^*(\pm 1)] \text{Tr} [K^\uparrow \varepsilon_{\downarrow}^*(\pm 1)], \\
\text{Tr} [P^\uparrow \varepsilon_{\downarrow}^*(\pm 1) K^\uparrow] &= \frac{\pm 1}{\sqrt{2}} \left(\text{Tr} [P^\uparrow \varepsilon_{\downarrow}^*(\pm 1)] \text{Tr} [K^\uparrow \varepsilon_{\downarrow}^*(0)] \right. \\
&\quad \left. + \text{Tr} [P^\uparrow \varepsilon_{\downarrow}^*(0)] \text{Tr} [K^\uparrow \varepsilon_{\downarrow}^*(\pm 1)] \right), \\
\text{Tr} [P^\uparrow \varepsilon_{\downarrow}^*(0) K^\uparrow] &= \frac{1}{\sqrt{6}} \left(-\text{Tr} [P^\uparrow \varepsilon_{\downarrow}^*(+1)] \text{Tr} [K^\uparrow \varepsilon_{\downarrow}^*(-1)] \right. \\
&\quad + 2\text{Tr} [P^\uparrow \varepsilon_{\downarrow}^*(0)] \text{Tr} [K^\uparrow \varepsilon_{\downarrow}^*(0)] \\
&\quad \left. - \text{Tr} [P^\uparrow \varepsilon_{\downarrow}^*(-1)] \text{Tr} [K^\uparrow \varepsilon_{\downarrow}^*(+1)] \right).
\end{aligned} \quad (33)$$

As a result the matrix elements can be written as

$$\begin{aligned}
\mathcal{M}(^1S_0) &= \frac{c_1}{4tt'} \text{Tr} \left[J_{1\downarrow}(\lambda_1, \lambda_3) \left\{ K_1^\uparrow K_{2\downarrow} J_2^\uparrow(\lambda_2, \lambda_4) \right. \right. \\
&\quad \left. \left. - J_2^\uparrow(\lambda_2, \lambda_4) K_{2\downarrow} K_1^\uparrow \right\} \right], \\
\mathcal{M}(^3P_0) &= \frac{c_2}{4tt'} \left(2H \text{Tr} \left[J_{1\downarrow}(\lambda_1, \lambda_3) J_2^\uparrow(\lambda_2, \lambda_4) \right] \right. \\
&\quad \left. - G \text{Tr} \left[J_{1\downarrow}(\lambda_1, \lambda_3) K_2^\uparrow \right] \text{Tr} \left[J_{2\downarrow}(\lambda_2, \lambda_4) K_1^\uparrow \right] \right), \\
\mathcal{M}(^3P_1) &= \frac{c_3}{4tt'} \left(t \text{Tr} \left[J_2^\uparrow(\lambda_2, \lambda_4) \left\{ \varepsilon_\downarrow^*(\lambda_R) J_1^\uparrow(\lambda_1, \lambda_3) K_{2\downarrow} \right. \right. \right. \\
&\quad \left. \left. - K_{2\downarrow} J_1^\uparrow(\lambda_1, \lambda_3) \varepsilon_\downarrow^*(\lambda_R) \right\} \right] \\
&\quad + t' \text{Tr} \left[J_1^\uparrow(\lambda_1, \lambda_3) \left\{ \varepsilon_\downarrow^*(\lambda_R) J_2^\uparrow(\lambda_2, \lambda_4) K_{1\downarrow} \right. \right. \\
&\quad \left. \left. - K_{1\downarrow} J_2^\uparrow(\lambda_2, \lambda_4) \varepsilon_\downarrow^*(\lambda_R) \right\} \right] \right), \\
\mathcal{M}(^3P_2) &= \frac{c_4}{8tt'} \left(2F \text{Tr} \left[J_1^\uparrow(\lambda_1, \lambda_3) \varepsilon_{\downarrow\downarrow}^*(\lambda_R) J_2^\uparrow(\lambda_2, \lambda_4) \right] \right. \\
&\quad + \text{Tr} \left[K_1^\uparrow \varepsilon_{\downarrow\downarrow}^*(\lambda_R) K_2^\uparrow \right] \text{Tr} \left[J_{1\downarrow}(\lambda_1, \lambda_3) J_2^\uparrow(\lambda_2, \lambda_4) \right] \\
&\quad - \text{Tr} \left[K_1^\uparrow \varepsilon_{\downarrow\downarrow}^*(\lambda_R) J_2^\uparrow(\lambda_2, \lambda_4) \right] \text{Tr} \left[J_{1\downarrow}(\lambda_1, \lambda_3) K_2^\uparrow \right] \\
&\quad \left. - \text{Tr} \left[K_2^\uparrow \varepsilon_{\downarrow\downarrow}^*(\lambda_R) J_1^\uparrow(\lambda_1, \lambda_3) \right] \text{Tr} \left[J_{2\downarrow}(\lambda_2, \lambda_4) K_1^\uparrow \right] \right), \\
\mathcal{M}(^1D_2) &= \frac{c_5}{4c_1} \text{Tr} \left[K_1^\uparrow \varepsilon_{\downarrow\downarrow}^*(\lambda_R) K_2^\uparrow \right] \mathcal{M}(^1S_0).
\end{aligned} \tag{34}$$

It turns out that this is the fastest method using the WvdW formalism, thanks to a fast matrix multiplication in FORTRAN90. The other method using the WvdW spinorial inner products merely serves as a cross-check.

For the evaluation of the matrix element squared or of the density matrices, summations over the helicities of external particles have to be made. However, the matrix element need not be evaluated for every helicity configuration. From the choice of the polarization vectors in [4] it can be shown that the complex conjugate of a matrix element is related to the matrix element with flipped helicities:

$$\begin{aligned}
\text{Spin-0 resonances : } \mathcal{M}(\{-\lambda_i\}) &= (\prod_i \lambda_i) \mathcal{M}^*(\{\lambda_i\}), \\
\text{Spin-1 resonance : } \mathcal{M}(\{-\lambda_i\}, -\lambda_R) &= (\prod_i \lambda_i) \mathcal{M}^*(\{\lambda_i\}, \lambda_R), \\
\text{Spin-2 resonances : } \mathcal{M}(\{-\lambda_i\}, -\lambda_R) &= (\prod_i \lambda_i) (-1)^{\lambda_R} \mathcal{M}^*(\{\lambda_i\}, \lambda_R).
\end{aligned} \tag{35}$$

Here again the λ_i 's are referring to the incoming and outgoing leptons and λ_R is the helicity of the resonance.

It should be noted that in the limit $m_e \rightarrow 0$, where the WvdW calculations simplify dramatically, we have analytically checked that the WvdW calculation gives the same result for the total matrix element squared as the massless parts of the cross sections in appendix A.

3 Density Matrices

When one considers the decay of a resonance into some final state X ,

$$e^+ e^- \rightarrow e^+ e^- R \rightarrow e^+ e^- X, \tag{36}$$

one can write for the amplitude

$$\mathcal{M} = \sum_{\lambda_R} \xi(P, W) \mathcal{A}_{\lambda_R} \mathcal{D}_{\lambda_R}. \tag{37}$$

\mathcal{A}_{λ_R} describes the (two-photon) production of a resonance with helicity λ_R . This is our previously discussed matrix element. \mathcal{D}_{λ_R} describes the decay of a resonance with helicity λ_R into the final state X . The quantity $\xi(P, W)$ represents the propagator of the resonance and numerical factors. The total matrix element still depends on the external four-momenta. No implicit integrations over outgoing momenta have been carried out.

The square of the total matrix element is then given by

$$\sum |\mathcal{M}|^2 = |\xi(P, M)|^2 \sum_{\lambda_R, \lambda'_R} \mathcal{A}_{\lambda_R \lambda'_R} \mathcal{D}_{\lambda_R \lambda'_R} = |\xi(P, M)|^2 \text{Tr}(\mathcal{A} \mathcal{D}^*). \quad (38)$$

Here \sum is the summation over the initial and final states and the helicities of the intermediate state. The quantities $\mathcal{A}_{\lambda_R \lambda'_R}$ and $\mathcal{D}_{\lambda_R \lambda'_R}$ are $\mathcal{A}_{\lambda_R} \mathcal{A}_{\lambda'_R}^*$ and $\mathcal{D}_{\lambda_R} \mathcal{D}_{\lambda'_R}^*$, summed over the helicities of all particles but the resonance. They are the density matrices for the production and decay of the resonance. For a spin J resonance the density matrix is formed by a $(2J + 1) \times (2J + 1)$ -matrix.

GaGaRes calculates for every generated event the normalized density matrix for the production of the resonance

$$\rho_{\lambda_R \lambda'_R} = \frac{\sum_{\lambda_i} \mathcal{A}(\lambda_i, \lambda_R) \mathcal{A}^*(\lambda_i, \lambda'_R)}{\sum_{\lambda_i, \lambda_R} |\mathcal{A}(\lambda_i, \lambda_R)|^2}, \quad (39)$$

where the summation in the numerator is over the electron and positron helicities and in the denominator the summation is over all helicities.

The density matrix for the production only depends on the production process and can therefore be calculated in GaGaRes. If one knows the mechanism for the decay of the resonance into a state X , also that decay density matrix could be incorporated in GaGaRes. This is beyond the scope of the present investigation.

From the definition of ρ one can immediately derive two properties of this density matrix:

- ρ is hermitian : $\rho = \rho^\dagger$;
- The trace of the density matrix equals 1 : $\text{Tr}(\rho) = 1$.

However, from the relation between complex conjugated matrix elements and spin flipped matrix elements (35) and the definition for the normalized density matrix (39), one can also derive an additional symmetry. For spin-1 resonances it reads

$$\rho_{\lambda_R \lambda'_R} = \rho_{-\lambda'_R - \lambda_R}. \quad (40)$$

$$\rho_{\lambda_R \lambda'_R} = (-1)^{\lambda_R + \lambda'_R} \rho_{-\lambda'_R - \lambda_R}. \quad (41)$$

This symmetry, the hermiticity and the fact that the trace is one, reduce strongly the number of independent elements in the density matrix. For a spin-1 resonance the density matrix has one real diagonal element and two complex off-diagonal elements, e.g. $\rho_{-1,1}$ and $\rho_{-1,0}$. For a spin-2 resonance one has two real diagonal elements and 6 independent complex off-diagonal elements. The fact that the diagonal elements are real follows from hermiticity.

In GaGaRes only the relations in (35) have been used to reduce the amount of calculations. However, checks have been performed that the density matrices are hermitian and that equations (40) and (41) hold.

The density matrices are computed for every event in the resonance rest frame (RRF), as this is the most natural scheme for the description of the resonance decay. The positive z -axis, the quantization axis, is defined by the boost direction. The Lorentz-transformation from the lab-frame to this RRF is performed by subsequently an azimuthal rotation, a polar rotation and a boost. The rotation brings the momentum of the resonance in the lab frame along the z -axis and the boost then brings the resonance to rest. From an experimental point of view this choice of polarization axis is also favoured as it is straightforward to reconstruct the boost direction from the detected final state X .

Some results on the calculation of density matrices using GaGaRes and a discussion thereof can be found in [9].

4 MC Generation Schemes

The parametrization of the external four-momenta in the lab-frame is given in (4). From now on we will use the following shorthand notations

$$\begin{aligned} s_i &\equiv \sin \theta_i, \\ c_i &\equiv \cos \theta_i. \end{aligned} \tag{42}$$

Additionally we introduce θ_{34} , the angle between the two outgoing leptons, given by

$$c_{34} \equiv \cos \theta_{34} = -c_3 c_4 - s_3 s_4 \cos(\phi_3 - \phi_4 + \pi). \tag{43}$$

The general expression for the cross section of a $2 \rightarrow 3$ process reads

$$\begin{aligned} \sigma = \int (2\pi)^4 \delta^{(4)}(p_1 + p_2 - p_3 - p_4 - P) \frac{1}{4\sqrt{(p_1 \cdot p_2)^2 - m_e^4}} \frac{1}{4} \sum |\mathcal{M}|^2 \\ \frac{d^3 \vec{p}_3}{(2\pi)^3 2E_3} \frac{d^3 \vec{p}_4}{(2\pi)^3 2E_4} \frac{d^3 \vec{P}}{(2\pi)^3 2E_R}. \end{aligned} \tag{44}$$

Performing the integration over \vec{P} and rewriting the remaining delta function gives

$$\begin{aligned} \sigma = \int \frac{1}{(2\pi)^5} \frac{\frac{1}{4} \sum |\mathcal{M}|^2}{8sE_3E_4 \sqrt{1 - \frac{4m_e^2}{s}}} \delta(4E_b^2 - 4E_b(E_3 + E_4) + 2m_e^2 + 2E_3E_4 \\ - 2|\vec{p}_3||\vec{p}_4| \cos \theta_{34} - M^2) d^3 \vec{p}_3 d^3 \vec{p}_4. \end{aligned} \tag{45}$$

Solving the delta function for E_3 yields

$$E_3 = \frac{-\xi(E_4 - 2E_b) \pm D}{2\{(E_4 - 2E_b)^2 - |\vec{p}_4|^2 c_{34}^2\}}, \tag{46}$$

where

$$\begin{aligned} \xi &= 4E_b^2 - 4E_bE_4 + 2m_e^2 - M^2, \\ D &= |\vec{p}_4| c_{34} |\tilde{\Delta}|, \\ \tilde{\Delta}^2 &= \xi^2 - 4(E_4 - 2E_b)^2 m_e^2 + 4|\vec{p}_4|^2 c_{34}^2 m_e^2. \end{aligned} \tag{47}$$

The sign ambiguity in (46) can be solved by dividing the phase-space into two parts. The cross sections of the two solutions are denoted by σ_{\pm} , corresponding to the sign in (46).

For the two solutions we find

- – solution

$$\begin{aligned}
E_{4,min} &= m_e, \\
E_{4,max} &= E_b - \frac{m_e M}{2E_b} - \frac{M^2}{4E_b}, \\
m_e \leq E_4 \leq E'_4 \text{ then } -1 \leq c_{34} \leq +1, \\
E'_4 \leq E_4 \leq E_{4,max} \text{ then } -1 \leq c_{34} \leq -c_a
\end{aligned} \tag{48}$$

- + solution

$$\begin{aligned}
E_{4,min} &= E'_4, \\
E_{4,max} &= E_b - \frac{m_e M}{2E_b} - \frac{M^2}{4E_b}, \\
-1 \leq c_{34} \leq -c_a,
\end{aligned} \tag{49}$$

where

$$E'_4 = \frac{4E_b^2 - 4m_e E_b + 2m_e^2 - M^2}{4E_b - 2m_e}, \tag{50}$$

and

$$c_a = \sqrt{\frac{4(2E_b - E_4)^2 m_e^2 - (4E_b^2 - 4E_4 E_b + 2m_e^2 - M^2)^2}{4|\vec{p}_4|^2 m_e^2}}. \tag{51}$$

Making a change of variables and performing the integration over E_3 leads to the total cross section

$$\sigma_{\pm} = \int \frac{\frac{1}{4} \sum |\mathcal{M}|^2}{256\pi^5 s} \frac{1}{\sqrt{1 - \frac{4m_e^2}{s}}} \frac{|\vec{p}_3| |\vec{p}_4|}{|4E_b - 2E_4 + 2\frac{|\vec{p}_4| E_3}{|\vec{p}_3|} c_{34}|} dE_4 d^2\Omega_3 d^2\Omega_4. \tag{52}$$

From now on all variables are normalized to the beam energy E_b . After this normalization the cross section reads

$$\sigma_{\pm} = \int \frac{\frac{1}{4} \sum |\mathcal{M}|^2}{256\pi^5 s} \frac{1}{\sqrt{1 - \frac{4m_e^2}{s}}} \frac{|\vec{p}_3| |\vec{p}_4|}{|4 - 2E_4 + 2\frac{|\vec{p}_4| E_3}{|\vec{p}_3|} c_{34}|} dE_4 d^2\Omega_3 d^2\Omega_4. \tag{53}$$

GaGaRes offers the user the choice between two schemes in which different sets of variables are generated to perform the MC integration. Scheme I, based on [10], is the most efficient scheme for no-tag events, as the peaking behaviour of the matrix element is better described. Scheme II is more efficient for single- or double-tag events, as both polar angles are generated variables.

Scheme I is described in [10] for the process

$$e^+ e^- \rightarrow e^+ e^- l^+ l^- \quad (l = e, \mu, \tau),$$

where the outgoing electron and positron are not tagged. As we have here one particle less in the final state, some simplifications can be introduced. This generation scheme being very similar to the scheme in [10], will not be discussed in this paper.

The second scheme is more efficient not only in the generation of single- and double-tagged events, but also for the generation of events with cuts on Q_i^2 , where an automatic cut on the angles is imposed.

The procedure is to start with the normalized expression given in (53). The first approximation one makes is to replace the expression for the matrix element

$$\frac{1}{4} \sum |\mathcal{M}|^2 \rightarrow \frac{64}{tt'}. \tag{54}$$

The corresponding weight factor is given by

$$W_1 = \frac{tt' \frac{1}{4} \sum |\mathcal{M}|^2}{64}. \quad (55)$$

In this approximation it is better to put a factor tt' rather than a factor $t^2t'^2$ in the denominator: from the formulae in appendix A it follows that in the limit where t and t' are of order m_e^2 , the numerator also is of order m_e^2 , for it can be shown that in this limit the invariant σ_{2-} is also proportional to m_e^2 . This reduces effectively the behaviour of the denominator to tt' . This approximation also gives the well-known logarithmic behaviour of the total cross section with respect to the total energy.

The cross section now reads

$$\sigma_{\pm} = \frac{1}{4\pi^5 s} \frac{1}{\sqrt{1 - \frac{4m_e^2}{s}}} \int \frac{1}{tt'} \frac{|\vec{p}_3||\vec{p}_4|}{|4 - 2E_4 + 2\frac{|\vec{p}_4|E_3}{|\vec{p}_3|}c_{34}|} dE_4 d^2\Omega_3 d^2\Omega_4. \quad (56)$$

Before performing the next approximations, let us consider the photon propagators more closely. The square of the invariant mass of the photon, radiated by the incoming positron, can be written as

$$t = -2E_3(1 - c_3 + \delta_3), \quad (57)$$

with

$$\delta_3 = \left(\frac{1 - |\vec{p}_3|P_b}{E_3} \right) c_3 - \frac{m_e^2}{E_3}. \quad (58)$$

Now $\delta_{3,min}$ is defined as

$$\delta_{3,min} \equiv \delta_3|_{c_3=1} = \frac{m_e^2(1 - E_3)^2}{2E_3^2}, \quad (59)$$

which reaches its minimum at $E_3 = E_{3,max} = 1 - \frac{1}{2}m_e M - \frac{1}{4}M^2$ and is denoted by

$$\varepsilon \equiv \delta_{3,min}|_{E_3=E_{3,max}} = \frac{m_e^2(m_e M + \frac{1}{2}M^2)^2}{8(1 - \frac{1}{2}m_e M - \frac{1}{4}M^2)^2}. \quad (60)$$

For t' we find a similar expression with the subscript 3 replaced by 4.

The second approximation now consists of the substitution

$$\frac{1}{tt'} \rightarrow \frac{1}{4E_3E_4(1 - c_3 + \varepsilon)(1 - c_4 + \varepsilon)}. \quad (61)$$

The corresponding weight factor reads

$$W_2 = \frac{4E_3E_4(1 - c_3 + \varepsilon)(1 - c_4 + \varepsilon)}{tt'}. \quad (62)$$

The third approximation consists of the approximation in the denominator

$$\frac{|\vec{p}_3||\vec{p}_4|}{E_3E_4} |2 - E_4 + \frac{|\vec{p}_4|E_3}{|\vec{p}_3|}c_{34}| \rightarrow 1 - E_4. \quad (63)$$

This approximation works rather well, as for most generated events the outgoing leptons go (almost) back to back ($c_{34} \approx -1$).

$$W_3 = \frac{p_3p_4}{E_3E_4} \frac{1 - E_4}{|2 - E_4 + \frac{|\vec{p}_4|E_3}{|\vec{p}_3|}c_{34}|}. \quad (64)$$

After these approximations the cross section reads

$$\sigma_{\pm} = \frac{1}{32\pi^5 s} \frac{1}{\sqrt{1 - \frac{4m_e^2}{s}}} \int \frac{dE_4 d^2\Omega_3 d^2\Omega_4}{(1 - c_3 + \varepsilon)(1 - c_4 + \varepsilon)(1 - E_4)}. \quad (65)$$

First, the trivial azimuthal integrations of ϕ_3 and ϕ_4 over $[\phi_{min}, \phi_{max}]$ can be performed. The integrations over the polar angles and over E_4 can be performed by using the standard integral

$$\int_{x_{min}}^{x_{max}} \frac{dx}{a - x} = \ln \left| \frac{a - x_{min}}{a - x_{max}} \right|. \quad (66)$$

The results of the integrations are put into the last two weights,

$$W_4 = \ln \left| \frac{1 + \varepsilon - c_{3,min}}{1 + \varepsilon - c_{3,max}} \right| \ln \left| \frac{1 + \varepsilon - c_{4,min}}{1 + \varepsilon - c_{4,max}} \right| (\phi_{3,max} - \phi_{3,min})(\phi_{4,max} - \phi_{4,min}), \quad (67)$$

$$W_{5,\pm} = \ln \left| \frac{1 - E_{4,min}}{1 - E_{4,max}} \right|,$$

while the “total cross section” is given by

$$\sigma_{tot} = \frac{1}{32\pi^5 s} \frac{1}{\sqrt{1 - \frac{4m_e^2}{s}}}. \quad (68)$$

The real total cross section can now be determined using

$$\sigma = \sigma_{tot} (< W_1 W_2 W_3 W_4 (W_{5+} + W_{5-}) >), \quad (69)$$

where $< W_1 W_2 W_3 W_4 (W_{5+} + W_{5-}) >$ is defined as the average of the total weight. During the generation of events there is a probability $W_{5-}/(W_{5+} + W_{5-})$ that an event is generated in the phase space of the $-$ solution. Especially for the heavier resonances almost all events will be generated in the $-$ part of the solution.

It is useful to give some comments on the generation of the integration variables. The variables are now generated by importance sampling in the following order: $\phi_3, \phi_4, \cos \theta_3, \cos \theta_4$. Let \mathcal{R} be a random variable uniformly distributed over the interval $[0, 1]$. Now the azimuthal angles ϕ_3 and ϕ_4 are generated according to

$$\begin{aligned} \phi_3 &= (\phi_{3,max} - \phi_{3,min})\mathcal{R} + \phi_{3,min}, \\ \phi_4 &= (\phi_{4,max} - \phi_{4,min})\mathcal{R} + \phi_{4,min}. \end{aligned} \quad (70)$$

Variables that are distributed like the integrand of (66) should be generated according to

$$x = a - (a - x_{max}) \left(\frac{a - x_{min}}{a - x_{max}} \right)^{\mathcal{R}}, \quad (71)$$

where it should be noted that in the program $1 - x$ rather than x is generated in order to guarantee the best numerical stability. At this point $\cos \theta_{34}$ can be evaluated. Then E_4 is generated. When $E_4 < E'_4$ one has only a contribution from the $-$ solution of the delta function. When $E_4 > E'_4$ one has to check whether c_{34} is in the allowed interval. If so, one has contributions from both solutions of the delta function.

Cuts on the generated variables are incorporated in the generation mechanism. They do not lead to a drop in efficiency (exactly the reason why scheme II is favoured for single- and double-tag reactions). For scheme II the program also offers the user the possibility

to impose cuts on E_3 , on the photon virtualities Q_i^2 and on c_{34} . The cuts are applied by the following replacement

$$|\mathcal{M}|^2 \rightarrow |\mathcal{M}|^2 \prod \theta, \quad (72)$$

where $\prod \theta$ is a product of step functions representing the cuts.

These cuts lead to a drop in generation efficiency. However, for the cuts on the photon virtualities Q_i^2 the angles of the corresponding polar angles are adapted in order to decrease this drop in efficiency. For a cut on the virtualities we use the following approximation formulae

$$\theta_{i,min} = 2\arcsin \sqrt{\frac{Q_{i,min}^2 + 2m_e^2}{4E_b E_{3,max}}}, \quad (73)$$

$$E_{3,min} = \frac{Q_{i,min}^2 + 2m_e^2}{4E_b \sin^2 \left(\frac{\theta_{3,max}}{2} \right)}, \quad (74)$$

where θ_i is the minimum polar angle of the outgoing lepton corresponding to the photon virtuality Q_i . On this photon virtuality the cut $Q_i > Q_{i,min}$ is imposed.

The average normalized density matrix is calculated by taking the weighted sum of the normalized density matrix and by dividing this sum by the total weight. This means that for every generated event in the lab system the momenta in the RRF are obtained which are then used to evaluate the RRF density matrix.

The program generates at the same time unweighted events, i.e. events with a weight factor equal to 1. This is done by applying a hit or miss algorithm.

5 Structure of the Program

A complete overview of all subroutines can be found in the code-listing of GaGaRes.

In the main program the file `gamgaminput` is read out. This file contains some input parameters. The variables that are read from `gamgaminput` are marked with an * in the following discussion.

5.1 Modules

A list of the most important modules in GaGaRes

`MODULE global_data`

In this module general variables are stored. Also some cuts and switches are contained in this module.

<code>Events*</code>	Number of events to be generated.
<code>StartNumber*</code>	Number of events to be generated to determine the maximum weight.
<code>IAcc</code>	Number of accepted events (after hit or miss integration).
<code>ICut</code>	Number of events after applied cuts.
<code>Eb*</code>	Beam energy.
<code>EWE*</code>	The expected maximum weight.
<code>Th3Min*,Th3Max*</code>	The minimum and maximum allowed polar angle of the scattered positron.

Th4Min*,Th4Max*	The minimum and maximum allowed polar angle of the scattered electron.
Ph3Min*,Ph3Max*	The minimum and maximum allowed azimuthal angle of the scattered positron.
Ph4Min*,Ph4Max*	The minimum and maximum allowed azimuthal angle of the scattered electron.
E3Min*,E3Max*	The minimum and maximum allowed energy of the scattered positron.
E4Min*,E4Max*	The minimum and maximum allowed energy of the scattered electron.
OutputFile*	The name of the file in which the generation information is written.
ICalc*	Determines which calculational method to use: =1 : Use the matrix element in terms of invariants (I); =2 : Use the WvdW formalism in terms of inner products (II); =3 : Use the WvdW formalism in terms of traces of matrix products(III); =4 : Use I and II; =5 : Use I and III; =6 : Use II and III; =7 : Use all methods.
IDens*	Switch variable (0,1). When set to 1 the density matrix is calculated.
ICut*	Switch variable (0,1). When set to 1 checks for cuts on non-generated variables are made. Note that this switch variable is <i>not</i> automatically set.
IBoost*	Switch variable (0,1). When set to 1 the resonance momentum in its rest frame is set exactly to $(M, 0, 0, 0)$ and ϕ_R and θ_R in this frame are set to 0.
It(p)Cut*	Switch variable (0,1). When set to 1 the cut on t (t') is taken into account and the minimum allowed scattering angle is adapted.
t(p)Min*,t(p)Max*	The minimum and maximum allowed values of t and t' .
IBreitWigner*	Switch variable (0,1). When set to 0 the invariant mass of the $\gamma\gamma$ -system, W , is set to M , the mass of the generated resonance. When set to 1, W is distributed according to a Breit-Wigner shape

$$\frac{1}{\pi} \frac{\Gamma}{(W-M)^2 + \Gamma^2},$$

where Γ is the total decay width of the resonance.

As the photons have a Bremsstrahlung-like character,
a low W is preferred (high weight). In order to
keep the possibility to generate unweighted events,
one can choose to generate the invariant
masses only in the interval $[M_R - n\Gamma, M_R + n\Gamma]$.

MODULE PhysCon

Some physical constants are stored in **PhysCon**. Between brackets their values are given (All masses and energies in GeV):

me	m_e ($= 0.00051099906$ GeV);
Alpha	$\alpha_{e.m.}$ ($= \frac{1}{137.036}$);
Pi	π ($= 3.14159$);
BarnConv	A conversion factor to go from GeV^{-2} to pb ($= 3.89385 \times 10^5 \text{ pbGeV}^2$).

MODULE Resonance

Contains some properties of the resonance.

IWav*	Determines the wave function of the resonance: $=1 : {}^1S_0; =2 : {}^3P_0; =3 : {}^3P_1; =4 : {}^3P_2; =5 : {}^1D_2;$
IRes*	Determines the composition of the resonance: $=1 : I = 0; =2 : I = 1; =3 : I = 1'; =4 : c\bar{c}; =5 : b\bar{b};$
RName	Contains the name of the resonance;
RMass	Contains the mass of the resonance;
RWf2	Contains the square (of the derivative) of the radial part of the wave function in the origin(see [1]);
eq2	Contains the value for e_q^2 (see [1]);
GGwidth	Contains the value of $\Gamma_{\gamma\gamma}$, the two-photon width;
Width	Contains the value of Γ , the total decay width of the resonance.

5.2 COMMON blocks

Some important COMMON blocks:

COMMON FourVecs

Contains the four-momenta of the generated particles in the lab-frame.

PV1	p_1 : four-momentum of incoming positron;
PV2	p_2 : four-momentum of incoming electron;
PV3	p_3 : four-momentum of outgoing positron;
PV4	p_4 : four-momentum of outgoing electron;
PVR	P_R : four-momentum of outgoing resonance.

COMMON BFourVecs

Contains the four-momenta in the rest frame of the resonance (RRF) where the positive z -axis is given by the boost direction $\hat{\beta}$. This COMMON block also contains the five four-momenta of the external particles.

COMMON WvdWSpinors

Contains the WvdW spinors of the external particles.

5.3 Subroutines

A collection of some important subroutines:

<code>InitPhysCon</code>	Initializes the physical constants;
<code>ResProp</code>	Initializes the resonance properties;
<code>MC</code>	Generates an event using scheme I;
<code>MC2</code>	Generates an event using scheme II;
<code>Boost</code>	Performs the boost to the RRF;
<code>StoreVar</code>	Stores the data of the generated event;
<code>Finish</code>	Calculates the cross sections after the generation loop and prints the results.

5.4 Functions

A collection of some important functions:

<code>RNF100</code>	Generates a random number uniformly distributed in the interval $[0, 1]$;
<code>MatrixElement2</code>	Calculates $\sum \mathcal{M} ^2$ using the expression in terms of invariants;
<code>SpinorMatrixElement2</code>	Calculates the matrix element squared (and the density matrix) using the WvdW calculation with spinor inner products;
<code>FastSpinMat2</code>	Calculates the matrix element squared (and the density matrix) using the WvdW calculation with traces of matrix products (preferred method for density matrix calculations);
<code>Norm</code>	Calculates the length of the spatial part of a four-vector.

6 Comparison to other MC generators

The results of the cross section calculations have been checked with the results of Galuga [11]. In this MC generator the same model is implemented in the Budnev [5] formalism. Within the errors we found complete agreement. Density matrix calculations could not be checked with any Monte Carlo generator, since only GaGaRes is able to calculate them at present. Only by extending the Galuga generator ourselves checks could be performed and gave agreement.

7 Conclusions

GaGaRes is a MC generator which is well-suited for the description of resonance production in two-photon physics. It offers several schemes and calculational methods and can be used for every required topology of the outgoing particles. The program is also able to generate density matrices which form an essential tool in the description of the decay of the resonances.

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A Expressions for $\Sigma |\mathcal{M}|^2$

In the expressions for the total matrix elements squared we have used the following variables to compactify the expressions

$$\begin{aligned}
\Sigma_{n\pm} &= s^n + s'^n \pm u^n \pm u'^n, \\
\sigma_{2\pm} &= ss' \pm uu', \\
s_{n\pm} &= s^n \pm s'^n, \\
u_{n\pm} &= u^n \pm u'^n, \\
t_{n\pm} &= t^n \pm t'^n,
\end{aligned} \tag{75}$$

with the convention that for $n = 1$ the n is not written.

This leads us to the following expressions:

1S_0 amplitude:

$$\begin{aligned}
\frac{1}{4} \Sigma |\mathcal{M}(^1S_0)|^2 &= \frac{c_1^2}{t^2 t'^2} \left\{ \frac{1}{2} \left[tt'(s_+^2 + u_+^2) - (tt' + \sigma_{2-})^2 - (tt' - \sigma_{2-})^2 \right] \right. \\
&\quad + m_e^2 \left[t_+(s_-^2 + u_-^2) + 4\sigma_{2-}\Sigma_- - 4tt'\Sigma_+ + 2s_-u_-t_- \right] \\
&\quad \left. + 16m_e^4 s_+ u_+ - 64m_e^6 \Sigma_+ + 256m_e^8 \right\}.
\end{aligned} \tag{76}$$

3P_0 amplitude:

$$\begin{aligned}
\frac{1}{4} \Sigma |\mathcal{M}(^3P_0)|^2 &= \frac{c_2^2}{t^2 t'^2} \left\{ \frac{B^2}{2} (2\sigma_{2-}^2 + tt'(s_-^2 + u_-^2 + 2tt')) + Btt'(-s_+(\sigma_{2-} + tt') \right. \\
&\quad + u_+(\sigma_{2-} - tt')) + 2t^2 t'^2 \sigma_{2+} + m_e^2 [B^2(t_+\Sigma_+^2 - 4\sigma_{2-}\Sigma_-) \\
&\quad + 2tt'B(\Sigma_-^2 - 2t_+\Sigma_+ + 4tt') - 4t^2 t'^2 (\Sigma_+ - t_+)] \\
&\quad \left. + m_e^4 [4B^2(\Sigma_-^2 - 4t_+\Sigma_+) + 32tt't_+B + 16t^2 t'^2] + 64m_e^6 B^2 t_+ \right\}.
\end{aligned} \tag{77}$$

Here B is given by

$$B = k_1 \cdot k_2 + W^2 = \frac{3}{2}(s + s' + u + u') + t + t' - 12m_e^2 = \frac{3}{2}\Sigma_+ + t_+ - 12m_e^2. \tag{78}$$

3P_1 amplitude:

$$\begin{aligned}
\frac{1}{4} \Sigma |\mathcal{M}(^3P_1)|^2 &= \frac{c_3^2}{W^2 t^2 t'^2} \left\{ tt' \left[tt't_+^2 - tt't_+\Sigma_+ + \frac{1}{2}t_{2+}\Sigma_{2+} - 2tt'\Sigma_{2+} \right. \right. \\
&\quad + (t_{2+} - 4tt')\sigma_{2+} + \sigma_{2-}\Sigma_{2-} - 4tt's_+u_+ + t_+\sigma_{2+}\Sigma_+ \\
&\quad + 2t(su'^2 + s^2u' + s'u^2 + s'^2u) + 2t'(su^2 + s^2u + s'u'^2 + s'^2u') \\
&\quad \left. \left. + 2t^2(su' + s'u) + 2t'^2(su + s'u') \right] - \sigma_{2-}^2 t_{2+} \right\}
\end{aligned} \tag{79}$$

$$\begin{aligned}
& +m_e^2 \left[t_{3+} \Sigma_+^2 - 7tt't_{+} \Sigma_{2+} - 26tt'(t(su' + s'u) + t'(su + s'u')) \right. \\
& + 8t_{2+}(ss's_+ + uu'u_+) + 4(t_{2+} + tt')(ss'u_+ + uu's_+) \\
& - 22tt't_{+} \sigma_{2+} - 12tt't_{2+} \Sigma_+ + 32t^2 t'^2 \Sigma_+ - 2tt' \Sigma_{3+} \\
& - 6tt'(ss's_+ + uu'u_+) + 16t^2 t'^2 t_+ + 2tt'(s_+ u_{2+} + u_+ s_{2+}) \\
& + 4t^2(su'^2 + s^2 u' + s' u^2 + s'^2 u) + 4t'^2(su^2 + s^2 u + s' u'^2 + s'^2 u') \\
& \left. - 10tt'(stu + st'u' + s't'u + s'tu') \right] \\
& + 16m_e^4 \left[-t_+^2 s_+ u_+ + 6tt't_{+} \Sigma_+ - t_{3+} \Sigma_+ - t_{2+}(\Sigma_{2+} + 4\sigma_{2+}) \right. \\
& + 4tt't_-^2 - 2(t^2(su' + s'u) + t'^2(su + s'u')) \left. \right] \\
& + 64m_e^6 [2(t_{2+} + tt') \Sigma_+ + t_{3+} - 5tt't_+] \\
& \left. - 256m_e^8 t_+^2 \right\}
\end{aligned}$$

3P_2 amplitude:

$$\begin{aligned}
\frac{1}{4} \sum |\mathcal{M}(^3P_2)|^2 = & \frac{c_4^2}{12} \left[\frac{1}{tt'} \left\{ \frac{m_e^2}{W^4} [-8\Sigma_+^3 - 32\sigma_{2+} \Sigma_+ + 64s_+ u_+ \Sigma_+ \right. \right. \\
& + 192(ss'u_+ + uu's_+)] + \frac{m_e^2}{W^2} [-4\Sigma_+^2 + 128\sigma_{2+} - 16s_+ u_+] \\
& + m_e^2 [-24\Sigma_+] + \frac{m_e^4}{W^4} [-192\Sigma_+^2 - 256\sigma_{2+} - 640s_+ u_+] \\
& + \frac{m_e^4}{W^2} [-128\Sigma_+] + m_e^4 [96] + \frac{m_e^6}{W^4} [3072\Sigma_+] + \frac{m_e^6}{W^2} [512] + \frac{m_e^8}{W^4} [-8192] \\
& + \frac{1}{W^4} [\Sigma_+^4 - 2s_+ u_+ (s_+^2 + u_+^2) + 16\sigma_{2-}^2 - 8s_+ u_+ \sigma_{2+} \\
& - 16(ss'u_{2+} uu's_{2+}) - 4s_{2+} u_{2+} - 48ss'uu'] + \frac{1}{W^2} [4\Sigma_+^3 - 8s_+ u_+ \Sigma_+ \\
& - 26\sigma_{2+} \Sigma_+ + 20(ss'u_+ + uu's_+)] + [s_+^2 + u_+^2 + 8\sigma_{2+}] \left. \right\} \\
& + \left(\frac{1}{tt'^2} + \frac{1}{t'^2 t'} \right) \left\{ \frac{m_e^2}{W^4} [2\Sigma_+^4 - 128\sigma_{2-}^2 - 32\sigma_{2-} \Sigma_{2-}] \right. \\
& + \frac{m_e^2}{W^2} [8\Sigma_+^3 + 8\sigma_{2-} \Sigma_-] + m_e^2 [2\Sigma_+^2] + \frac{m_e^4}{W^4} [-32\Sigma_+^3 - 128s_+ u_+ \Sigma_+ \\
& + 256\sigma_{2-} \Sigma_-] + \frac{m_e^4}{W^2} [-200\Sigma_+^2 + 32s_+ u_+] + m_e^4 [64\Sigma_+] \\
& + \frac{m_e^6}{W^4} [512\Sigma_+^2 + 1024s_+ u_+] + \frac{m_e^6}{W^2} [1536\Sigma_+] + m_e^6 [-256] \\
& \left. + \frac{m_e^8}{W^4} [-4096\Sigma_+] + \frac{m_e^8}{W^2} [-4096] + \frac{m_e^{10}}{W^4} [8192] \right\}
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{W^4} [8\sigma_{2-}^2 \Sigma_+] + \frac{1}{W^2} [-2\sigma_{2-}^2] \Big\} + \frac{m_e^2}{t'^2} [-24(s+u)(s'+u')] \\
& + \frac{m_e^2}{t^2 t'} [-24(s+u')(s'+u)] + \frac{1}{t} \Big\{ \frac{m_e^2}{W^4} [8\Sigma_+^2 + 64(s+u')(s'+u) \\
& + 128(su' + s'u) + 32t'\Sigma_+] + \frac{m_e^2}{W^2} [-64\Sigma_+ - 48t'] + m_e^2 [48] \\
& + \frac{m_e^4}{W^4} [-640\Sigma_+ - 256t'] + \frac{m_e^4}{W^2} [416] + \frac{m_e^6}{W^4} [2560] + \frac{1}{W^4} [2\Sigma_+^3 \\
& + 2t'\Sigma_+^2 - 8t'(s+u)(s'+u') - 24(ss'u_+ + uu's_+) - 8(su'(s+u') \\
& + s'u(s'+u))] + \frac{1}{W^2} [4\Sigma_+^2 - 4(s+u)(s'+u') - 12\sigma_{2+}] \Big\} \\
& + \frac{1}{t'} \Big\{ \frac{m_e^2}{W^4} [8\Sigma_+^2 + 64(s+u)(s'+u') + 128(su + s'u') + 32t\Sigma_+] \\
& + \frac{m_e^2}{W^2} [-64\Sigma_+ - 48t] + m_e^2 [48] \\
& + \frac{m_e^4}{W^4} [-640\Sigma_+ - 256t] + \frac{m_e^4}{W^2} [416] + \frac{m_e^6}{W^4} [2560] + \frac{1}{W^4} [2\Sigma_+^3 \\
& + 2t\Sigma_+^2 - 8t(s+u')(s'+u) \\
& - 24(ss'u_+ + uu's_+) - 8(su(s+u) + s'u'(s'+u'))] \\
& + \frac{1}{W^2} [4\Sigma_+^2 - 4(s+u')(s'+u) - 12\sigma_{2+}] \Big\} \tag{80} \\
& + \frac{1}{t^2 t'^2} \Big\{ \frac{m_e^2}{W^4} [-56\sigma_{2-}^2 \Sigma_+ - 8\sigma_{2-} \Sigma_{3-} - 8\sigma_{2-} (s_+ u u' - u_+ s s')] \\
& + 8\sigma_{2-} (s_+ u_{2+} - u_+ s_{2+})] + \frac{m_e^2}{W^2} [-32\sigma_{2-}^2 - 8\sigma_{2-} \Sigma_{2-}] + m_e^2 [16\sigma_{2-} \Sigma_-] \\
& + \frac{m_e^4}{W^4} [416\sigma_{2-}^2 + 160\sigma_{2-} \Sigma_{2-} + 8\Sigma_{2+}^2 - 32s_{2+} u_{2+}] + \frac{m_e^4}{W^2} [8\Sigma_+^3 + \sigma_{2-} \Sigma_- \\
& - 32(s_+ u_{2+} + u_+ s_{2+}) - 64(ss'u_+ + uu's_+)] + m_e^4 [8\Sigma_+^2 + 64s_+ u_+] \\
& + \frac{m_e^6}{W^4} [-128\Sigma_+^3 - 512\sigma_{2-} \Sigma_- + 512s_+ u_+ \Sigma_+] + \frac{m_e^6}{W^2} [-64\Sigma_-^2] \\
& + m_e^6 [-384\Sigma_+] + \frac{m_e^8}{W^4} [512\Sigma_-^2] + m_e^8 [1536] + \frac{1}{W^4} [2\sigma_{2-}^2 \Sigma_{2+} + 4\sigma_{2-}^2 \sigma_{2+} \\
& + 4\sigma_{2-}^2 s_+ u_+] + \frac{1}{W^2} [2\sigma_{2-}^2 \Sigma_+] + [-4\sigma_{2-}^2] \Big\} \\
& + \left(\frac{1}{t^2} + \frac{1}{t'^2} \right) \Big\{ \frac{m_e^2}{W^4} [8\Sigma_+^3 - 32\sigma_{2-} \Sigma_-] + \frac{m_e^2}{W^2} [4\Sigma_+^2] + \frac{m_e^4}{W^4} [-160\Sigma_+^2 \\
& - 128s_+ u_+] + \frac{m_e^4}{W^2} [-160\Sigma_+] + \frac{m_e^6}{W^4} [1536\Sigma_+] + \frac{m_e^6}{W^2} [640] + \frac{m_e^8}{W^4} [-4096] \\
& + \frac{1}{W^4} [8\sigma_{2-}^2] \Big\} + \frac{1}{t^2} \Big\{ \frac{m_e^2}{W^4} [8t'\Sigma_+^2] + \frac{m_e^2}{W^2} [24(s+u')(s'+u)] + \frac{m_e^4}{W^4} [-128t'\Sigma_+] \\
& + \frac{m_e^6}{W^4} [512t'] \Big\} + \frac{1}{t'^2} \Big\{ \frac{m_e^2}{W^4} [8t\Sigma_+^2] + \frac{m_e^2}{W^2} [24(s+u)(s'+u')] + \frac{m_e^4}{W^4} [-128t\Sigma_+]
\end{aligned}$$

$$\begin{aligned}
& + \frac{m_e^6}{W^4} [512t] \Big\} + \Big\{ \frac{m_e^2}{W^4} [64\Sigma_+ + 32t_+] + \frac{m_e^2}{W^2} [-144] + \frac{m_e^4}{W^4} [-512] + \frac{1}{W^4} [2\Sigma_-^2] \\
& + \frac{1}{W^2} [12\Sigma_+ + 6t_+] + [8] \Big\} \Big].
\end{aligned}$$

1D_2 amplitude:

$$\frac{1}{4} \sum |\mathcal{M}(^1D_2)|^2 = \hat{c}^2 \mathcal{F} \frac{1}{4} \sum |\mathcal{M}(^1S_0)|^2, \quad (81)$$

with

$$\hat{c} = \frac{c_5}{c_1}, \quad (82)$$

and

$$\mathcal{F} = \frac{1}{2} \left(-t + \frac{(W^2 + t_-)^2}{4W^2} \right) \left(-t' + \frac{(W^2 - t_-)^2}{4W^2} \right) + \frac{1}{6} \left(\frac{W^2 - t_+}{2} - \frac{W^4 - (t_-)^2}{4W^2} \right)^2. \quad (83)$$

Note that for the 1D_2 case the overall factor $\frac{\epsilon^4}{t^2 t'^2}$ is included in the 1S_0 factor.

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